Canonical phase-space approach to the noisy Burgers equation

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(Received 8 December 1998; revised manuscript received 20 May 1999)

Presenting a general phase-space approach to stochastic processes we analyze in particular the Fokker-Planck equation for the noisy Burgers equation and discuss the time-dependent and stationary probability distributions. In one dimension we derive the long-time skew distribution approaching the symmetric stationary Gaussian distribution. In the short-time regime we discuss heuristically the nonlinear soliton contributions and derive an expression for the distribution in accordance with the directed polymer-replica model and asymmetric exclusion model results. [S1063-651X(99)00710-2]

PACS number(s): 05.10.Gg, 64.60.Ht, 05.45.Yv

The strong coupling aspects of systems driven stochastically far from equilibrium present a formidable challenge in modern statistical physics and soft-condensed matter. The phenomena in question are ubiquitous and include turbulence in fluids, interface and growth problems, chemical reactions, self-organized critical systems, and even economical and sociological models.

In recent years much of the focus of modern statistical physics and soft-condensed matter has shifted towards such systems. Drawing on the case of static and dynamic critical phenomena in and close to equilibrium where scaling, critical exponents, and universality have served to organize our understanding and to provide calculational tools, a similar approach has been advanced towards nonequilibrium phenomena with the purpose of elucidating scaling properties and more generally the morphology or pattern formation in a driven state.

Whereas perturbative field theory together with the dynamic renormalization group have proven successful in the context of dynamic critical phenomena, the extension to nonequilibrium phenomena is often plagued with both technical and conceptual problems. This is related to the occurrence of strong coupling features encountered, for example, in the notable case of hydrodynamical turbulence, and there is a need for the development of appropriate nonperturbative theoretical tools in order to access the strong coupling regime.

In this context the noisy Burgers equation for the slope $u_n=\nabla_n h$ $(n=1,\dots,d)$ of a growing interface [1],

$$\frac{\partial u_n}{\partial t}=\nu\nabla^2 u_n+\lambda u_p\nabla_p u_n+\nabla_n\eta, \tag{1}$$

or, equivalently, the Kardar-Parisi-Zhang (KPZ) equation [2] for the height h, $\partial h/\partial t=\nu\nabla^2 h+(\lambda/2)\nabla_n h\nabla_n h+\eta$, provide maybe the simplest continuum description of an open driven nonlinear system exhibiting strong coupling features such as pattern formation and a new dynamical scaling univerality class. In Eq. (1) ν is a damping constant or viscosity characterizing the linear diffusive term, λ is a coupling strength for the nonlinear mode coupling or growth term, and η is a Gaussian white noise driving the system into a stationary state and correlated according to $\langle\eta(x_n,t)\eta(x_n',t')\rangle=\Delta\Pi_n\delta(x_n-x_n')\delta(t-t')$, characterized by the noise strength Δ.

Notwithstanding the simple form of Eq. (1), the driven Burgers equation introduced originally in order to model aspects of turbulence and the KPZ equation providing the simplest description of a growing interface, the morphology and scaling properties embodied in Eq. (1) have been difficult to extract and a full understanding of Eq. (1) remains one of the important issues in nonequilibrium statistical physics [3]. The status is that besides perturbation theory in λ [4] that regarding the scaling properties provides the roughness and dynamic exponents $(\zeta,z)=(1/2,3/2)$ in $d=1$, but, otherwise, is limited to an ϵ expansion about the (lower) critical dimension $d=2$, yielding a kinetic phase transition above $d=2$ separating a weak-coupling phase (the $\lambda=0$ universality class) with exponents $(\zeta,z)=[(2-d)/2,2]$ from a strong-coupling phase and to all orders in ϵ the exponents $(\zeta,z)=(0,2)$ on the phase line [5], nonperturbative methods include (i) in the $d=1$ case mapping to spin models [6] and information gained from lattice models [7], (ii) mapping to directed polymers in combination with replica methods [3], (iii) mode-coupling expansions [8], and, most recently, (iv) operator expansions yielding the strong-coupling exponents $(\zeta,z)=(2/5,8/5)$ in $d=2$ and $(\zeta,z)=(2/7,12/5)$ in $d=3$, corresponding to skewness in the height distribution [9].

In a recent series of papers [10] we addressed the better understood one-dimensional case and advanced a nonperturbative approach to the noisy Burgers equation, which purports to elucidate both the morphology and scaling properties of a growing interface in $d=1$. Arguing that the noise strength Δ is the relevant nonperturbative parameter driving the system into a stationary state and thus circumventing (a) the limitations of perturbation theory which is based on an effective expansion in $\lambda^2\Delta/\nu^3$ assuming regularity in Δ [4] and (b) the self-consistency assumptions underlying the mode-coupling approach [8]; the method was based on a weak noise saddle-point approximation to the Martin-Siggia-Rose functional formulation [11] of the noisy Burgers equation (1). Importantly, the method yields coupled deterministic field equations for the slope u and a noise field φ

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(characterizing η), replacing the stochastic Burgers equation, admitting soliton solutions, and as a result a many-body formulation of the pattern formation of a growing interface in terms of a dilute gas of propagating solitons with superposed linear diffusive modes. The canonical form of the approach also (1) yields the soliton dispersion, $E \propto \lambda p^z$, $z=3/2$, and damped propagating linear mode dispersion $\omega = -i\nu k^2 + \lambda u k$, where u is the soliton amplitude [12], (2) recovers the scaling exponents $(\zeta, z) = (1/2, 3/2)$ and an expression for the scaling function, and (3) associates the Burgers universality class with the leading gapless soliton excitation.

In the present paper we develop a general canonical phase-space approach to a stochastic Langevin equation of the Burgers type with additive white noise. This method, which emerged from our previous studies alluded to above, allows us to discuss and in some cases derive the stationary and time-dependent weak-noise solutions of the associated Fokker-Planck equation for the probability distributions. In particular, for the Burgers equation (1) the time-dependent and stationary distributions are given by

$$P(u_n, T) \propto \exp\left[-\frac{1}{\Delta} S(u_n, T)\right], \tag{2}$$

$$P_{\rm st}(u_n) \propto \lim_{T\to\infty} \exp\left[-\frac{1}{\Delta} S(u_n, T)\right], \tag{3}$$

where the action has the canonical (symplectic) form

$$S = \int_0^T d^d x\, dt \left(p_n \frac{\partial u_n}{\partial t} - \mathcal{H}\right), \tag{4}$$

with Hamiltonian density $\mathcal{H} = p_n[\nu \nabla^2 u_n + \lambda u_m \nabla_m u_n - (1/2)\nabla_n \nabla_m p_m]$, yielding the coupled Hamiltonian equations of motion,

$$\left(\frac{\partial}{\partial t} - \lambda u_m \nabla_m\right) u_n = \nu \nabla^2 u_n - \nabla_n \nabla_m p_m, \tag{5}$$

$$\left(\frac{\partial}{\partial t} - \lambda u_m \nabla_m\right) p_n = -\nu \nabla^2 p_n + \lambda (p_n \nabla_m u_m - p_m \nabla_n u_m). \tag{6}$$

The above *mean field* or *hydrodynamical* equations, replacing the noisy Burgers equation (1), allow an analysis of the time-dependent distribution $P(u_n, T)$. In principle, we have to solve Eqs. (5) and (6), determining the orbits in $p_n u_n$ phase space, and compute the action (4) and thus $P(u_n, T)$ according to Eq. (2).

The general framework is developed along the following lines, see also Ref. [13]: The Fokker-Planck equation pertaining to a general Langevin equation with additive noise for the stochastic variable q_i (i is a discrete and/or continuous index),

$$\frac{dq_n}{dt} = -\frac{1}{2} F_n(q_l) + \eta_n, \tag{7}$$

$$\langle \eta_n(t) \eta_m(t') \rangle = \Delta K_{nm} \delta(t-t'), \tag{8}$$

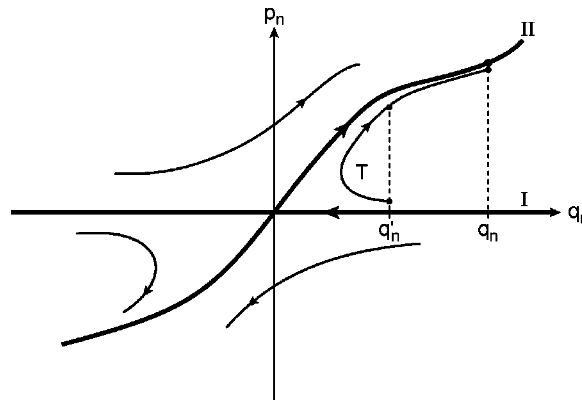

FIG. 1. Canonical phase space in the general case. The solid curves indicate the zero-energy *transient* submanifold (I) and *stationary* submanifold (II). The stationary saddle point is at the origin. The finite time (T) orbit from q'_n to q_n migrates to the zero-energy submanifold for $T \to \infty$.

has the form, denoting $\nabla_n = \partial/\partial q_n$, $\partial P/\partial t = (1/2)\nabla_n[\Delta K_{nm} \nabla_m P + F_n P]$. Searching for a solution of the form $P \propto \exp[-S/\Delta]$, it is an easy task to show that to leading order in the noise strength Δ, the action $S(q_n, t)$ satisfies the Hamilton-Jacobi equation $\partial S/\partial t + H(q_n, \nabla_n S) = 0$, where the energy $E = H$ and the canonically conjugate momentum $p_n = \nabla_n S$. The Hamiltonian H has the general form

$$H = (1/2)(K_{nm} p_n p_m - F_n p_n), \tag{9}$$

implying the Hamiltonian equations of motion,

$$\frac{dq_n}{dt} = K_{nm} p_m - \frac{1}{2} F_n, \tag{10}$$

$$\frac{dp_n}{dt} = \frac{1}{2} p_m \nabla_n F_m. \tag{11}$$

Assuming for simplicity that $F_n \to 0$ for $q_n \to 0$ the energy surfaces have the characteristic submanifold structure depicted in Fig. 1.

The origin in phase space constitutes a hyperbolic stationary point defined by the unstable zero-energy submanifold $p_n = 0$, the *transient manifold*, and, assuming the existence of a stationary state, a stable submanifold defined by $K_{nm} p_m - F_n$ orthogonal to p_n, the *stationary manifold*. The stationary state is determined by orbits on the zero-energy manifolds whose structure thus characterizes the nature of the stochastic problem. The action S and hence the distribution P are given by

$$S(q_n, T, q'_n) = \int_0^T dt \left[p_n \frac{dq_n}{dt} - H\right], \tag{12}$$

$$P(q_n, T, q'_n) \propto \exp[-S(q_n, T, q'_n)/\Delta], \tag{13}$$

where the orbit from q'_n to q_n is traversed in time T. The stationary distribution $P_{\rm st}(q_n)$ is thus obtained in the limit $T \to \infty$ and $E = 0$, assuming that $E(T) \propto \exp[-{\rm const} \times T]$ for $T \to \infty$,

$$P_{\rm st}(q_n)=\exp\left[-\frac{1}{\Delta}\int_0^\infty dt p_n \frac{dq_n}{dt}\right], \quad (14)$$

i.e., $P_{\rm st}$ is determined by the *infinite-time orbits on the zero-energy manifold*.

This structure of phase space permits a simple nonstochastic, deterministic discussion of the approach to the stationary state of a damped noise-driven system in terms of dynamical system theory. Referring to Fig. 1, consider an orbit from q_n' to q_n on the energy surface $E(T)$ traversed in time T. In order to attain the stationary state $E(T)\to 0$ in the limit $T\to\infty$. For $E\sim 0$ the initial part of the orbit moves close to the $p_n=0$ submanifold and from Eq. (10) is determined by $dq_n/dt=-(1/2)F_n$, i.e., the deterministic noiseless version of Langevin equation (7). In the absence of noise the motion is transient and damped. The orbit slows down near the stationary point (the origin in phase space) before it picks up again and moves close to the other submanifold $K_{nm}p_m-F_n\perp p_n$. This final part of the orbit terminating in q_n at time T thus corresponds to the establishment of the stationary state. Ergodic behavior, i.e., the independence of the initial configuration, is associated with the long (infinite) *waiting time* near (at) the stationary point.

In the special case $K_{nm}=\delta_{nm}$ and $F_n=\nabla_n\Phi$, corresponding to an effective fluctuation-dissipation theorem and an underlying free energy Φ, the energy (9) and the equations of motion (10) and (11) are consistent with the zero-energy submanifolds $p_n=0$ and $F_n=p_n$ and yield the equilibrium distribution $P_{\rm st}(q_n)\propto\exp[-\Phi/\Delta]$. In the general case a determination of the time-dependent and stationary distributions require a knowledge of the energy submanifolds in combination with a solution of the canonical equations in order to determine the orbits.

The noisy Burgers equation (1) falls within the scope of the general framework summarized above. With the identification $q_i(t)\to u_n(x_m,t)$, $K_{nm}\to\nabla^2\Pi_m\delta(x_m-x_m')$, and $F_n\to -2(\nu\nabla^2u_n+\lambda u_m\nabla_m u_n)$, we obtain Eqs. (2)–(6). Note that the canonical momentum, the *noise field* p_n, is essentially a *slaved variable*.

In the case $d=1$, which is the basis for our discussion here, the *mean field* equations (5) and (6) reduce to the form

$$\left(\frac{\partial}{\partial t}-\lambda u\nabla\right)u=\nu\nabla^2u-\nabla^2p, \quad (15)$$

$$\left(\frac{\partial}{\partial t}-\lambda u\nabla\right)p=-\nu\nabla^2p. \quad (16)$$

The distribution (2) is determined by the action (4), which has the general structure $S=S_{\rm st}(u)+S_{\rm skew}(u,T)+S_{sol}(u,T)$

The *stationary distribution* $P_{\rm st}(u)$ given by $S_{\rm st}(u)$ is easily found by noting that Eqs. (15) and (16) coincide on the zero-energy submanifold $p=2\nu u$ (the energy density $\mathcal{H}$ becomes a total derivative yielding $E=0$). Inserting $p=2\nu u$ and $E=0$ in Eqs. (2) and (4) in the limit $T\to\infty$ and integrating over time we obtain the symmetric Gaussian stationary distribution [14],

$$P_{\rm st}(u)\propto\exp\left[-(\nu/\Delta)\int dx u(x)^2\right]. \quad (17)$$

Also, setting $p=2\nu(u+\delta u)$ we find to leading order in $\delta u(\partial/\partial t-\lambda u\nabla)\delta u=\nu\nabla^2\delta u$. Noting that $\partial/\partial t-\lambda u\nabla$ is invariant under the Galilean transformation: $x\to x-\lambda u_0 t$, $u\to u+u_0$, and choosing an instantaneous frame with vanishing u, $\delta u\propto\exp[-\nu k^2 t]$, implying that the orbits approach the zero-energy stationary submanifold $p=2\nu u$.

The *long-time skew distribution* $P_{\rm skew}(u,T)$ is determined by $S_{\rm skew}(u,T)$. For $\lambda=0$ we obtain in wave-number space the symmetric contribution, $u_k=\int dx\exp(-ikx)u(x)$, i.e., the Gaussian or harmonic approximation,

$$S^0_{\rm skew}(u_k,T)=-\nu\int\frac{dk}{2\pi}|u_k|^2\exp[-2\nu k^2T], \quad (18)$$

defining a crossover time $T_{\rm co}\propto 1/\nu k^2$. For a finite system $k\propto 1/L$ (L is the system size), i.e., $T_{\rm co}\propto L^2/\nu$, yielding the dynamic exponent $z=2$ in accordance with the diffusive mode contribution.

For $\lambda\neq 0$ and for large T approximating the orbit *close to the manifold* by an orbit *on the manifold*, inserting $p=2\nu u$ in Eq. (15), we obtain the deterministic Burgers equation with $\eta=0$ and viscosity $-\nu$ that can be solved by means of the Cole-Hopf transformation [2], see also Refs. [15]. Thus setting $u=\nabla h$ and $h=-(2\nu/\lambda)\ln w$ yields the diffusion equation $\partial w/\partial t=-\nu\nabla^2 w$ for w, which is solved by means of the Green's function $G_x(T)=[4\pi\nu T]^{-1/2}\exp[-x^2/4\nu T]$. We obtain $S_{\rm skew}(u,T)=\nu\int dx u'(x)^2$, where $u=\nabla h$ and $u'=\nabla h'$ are related according to the nonlinear expression,

$$\exp[-(\lambda/2\nu)h']=\int dx' G_{x-x'}(T)\exp[-(\lambda/2\nu)h], \quad (19)$$

giving rise to the distribution

$$P(u,T)\propto P_{\rm st}(u)\exp\left[\frac{\nu}{\Delta}\int dx u'(x)^2\right], \quad (20)$$

which by inspection is skew. To order λ in wave-number space we also have, setting $G_{k,T}=\exp[-\nu k^2T]$,

$$P(u_k,T)\propto P_{\rm st}(u_k)P^0_{\rm skew}(u_k,T)P^\lambda_{\rm skew}(u_k,T), \quad (21)$$

where $P^0_{\rm skew}\propto\exp[-S^0_{\rm skew}/\Delta]$, $P^\lambda_{\rm skew}\propto\exp[-S^\lambda_{\rm skew}/\Delta]$, and the anharmonic (or field theoretical one-loop) expressions

$$S^\lambda_{\rm skew}(u_k,T)=2\lambda\int\frac{dk}{2\pi}\frac{dk'}{2\pi}F_{k,k',T},$$

$$F_{k,k',T}=[G_{k,2T}-G_{k,T}G_{k+k',T}G_{k',T}]u_k u_{-k-k'}h_{k'}, \quad (22)$$

exhibiting skewness.

The *short-time distribution* $P_{\rm sol}(u,T)$ given by $S_{\rm sol}(u,T)$, corresponding to an orbit off the zero-energy manifold, is determined by the soliton-diffusive mode contribution discussed in [10]. For a single soliton with boundary values u_+ and u_- the propagation velocity v is given by the soliton condition

$$u_++u_-=-2v/\lambda. \quad (23)$$

However, only the *left-hand* soliton ($u_+ < u_-$) carries nonvanishing energy, momentum, and action according to the assignment $E = (2/3)\nu\lambda(u_+^3 - u_-^3)$, $\Pi = \nu(u_+^2 - u_-^2)$, and $S = (1/6)\nu\lambda T|u_+ - u_-|^3$ (note the Galilean invariance of S). The action of a multisoliton configuration constituting a growing interface is then given by

$$S_{\rm sol}(u,T) = \frac{1}{6}\lambda\nu T \sum_{\rm lhs} |u_+ - u_-|^3, \tag{24}$$

where summation is over *left-hand* solitons (lhs) only.

Owing to the constraint imposed by the soliton condition (23) and the nonintegrability of the equation of motion we can only give a qualitative discussion of $P_{\rm sol}$. Since the saturation width of an interface is a finite-size effect time-scale separation only occurs for a finite system. Noting that the propagation of solitons and the imposition of periodic (or bouncing) boundary conditions, in order to ensure growth in h, endows the velocity with a scale, i.e., $v \sim L/T$, we obtain, inserting $u_+ + u_- = -2v/\lambda \sim u_+ - u_-$ in Eq. (24) and from $P_{\rm sol} \sim \exp[-S_{\rm sol}/\Delta]$ the soliton crossover time $T_{\rm co}^{\rm sol} \propto (1/\lambda)(\nu/\Delta)^{1/2} L^{3/2}$, which is also consistent with the dimensionless argument $\lambda(\Delta/\nu)^{1/2} t/x^{3/2}$ in the scaling function for the slope correlations discussed in [8,10].

In the short-time regime $T \ll T_{\rm co}^{\rm sol}$ the soliton configurations contribute to P. Noting that $|u_+ - u_-|^3 \sim (uL)^{3/2}(T\lambda)^{-3/2} \sim h^{3/2}(T\lambda)^{-3/2}$, we obtain, inserting in Eq. (24) (h is measured relative to the mean height),

$$P_{\rm sol}(h,T) \propto \exp[-(\nu/\Delta)(1/\lambda T)^{1/2} h^{3/2}], \tag{25}$$

in accordance with the directed polymer-replica-based result [3] and the exact results for the asymmetric exclusion model [16]. The skewness of the distribution then arises from the bias in the statistical weight $\exp[-S/\Delta]$ assigned to the *left*- and *right-hand* solitons giving rise to a predominance of *right-hand* solitons ($S=0$), corresponding to relative forward growth. In the long-time regime, $T \gg T_{\rm co}^{\rm sol}$, the soliton contribution vanishes and only the diffusive modes and their interactions contribute to P.

In this paper we have outlined a canonical phase-space approach to Langevin equations with additive noise; details can be found in [17]. In addition to providing insight from the dynamical system theory the method also yields a calculational tool for the determination of the weak-noise probability distributions. In particular, we have applied the method to the noisy Burgers equation in $d=1$ and derived expressions for the skew finite-time probability distribution. In the short-time regime our heuristic result is in agreement with the directed polymer-replica method.

Discussions with M. Kosterlitz, T. Hwa, J. Hertz, P. Cvitanović, K. B. Lauritsen, and A. Svane are gratefully acknowledged.

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